Q Algorithms

Input models

		Examples
Standard	$x = (x_1, \ldots, x_n) \in \{0,1\}^n$	factoring, 3-SAT
oracle	$f : [N] \to \{0,1\}$ $O_f \|x\rangle\|y\rangle = \|x\rangle\|y \oplus f(x)\rangle$	Grover HSP Ham simulation
quantum input	$\|\psi\rangle$	QFT
quantum oracle	U	phase estimation

linear systems

Quick review

Grover $= OR(f(1), \ldots, f(N))$

Alternate $O_f (I \otimes \ket{-})$ and $R_s = I - 2|s\rangle\langle s|$

$W = -R_s O_f$ $\qquad |s\rangle = \underbrace{\sin\theta}_{N^{-1/2}} |g\rangle + \cos\theta |b\rangle$

W rotates by 2θ

Variants

Ampl Ampl $\qquad$ M solutions $\Rightarrow \sin\theta = \sqrt{M/N}$

can amplify success prob of any algo

Approx Counting

need to estimate $\theta = \sin^{-1}\sqrt{p}$

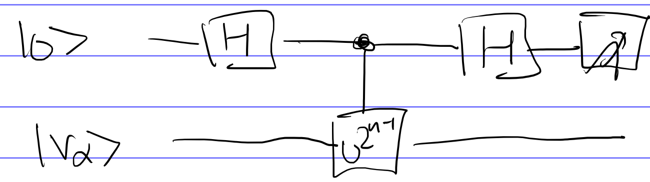
use QPE $\qquad$ phase estimation

QPE

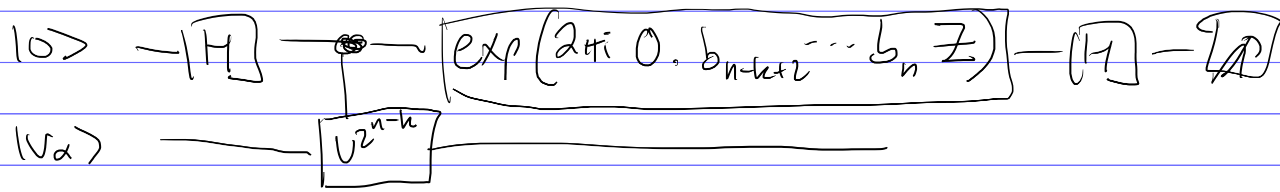
$$U = \sum_\alpha e^{i\theta_\alpha} |v_\alpha\rangle\langle v_\alpha|$$

suppose we can perform $C\text{-}U^{2^k}$

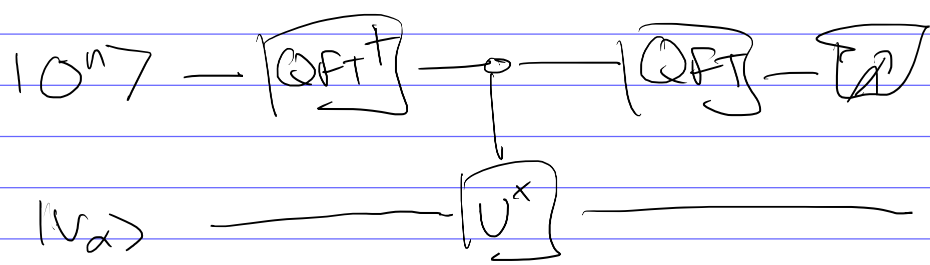
given input $|v_\alpha\rangle$ suppose $\frac{\theta_\alpha}{2\pi} = 0.b_1 b_2 b_3 \cdots \underbrace{b_n \cdots}_{\text{assume } 0}$



$$|+\rangle \Rightarrow \frac{|0\rangle + (-1)^{b_n} |1\rangle}{\sqrt{2}} \Rightarrow |b_n\rangle$$



Equivalently



$$QFT\,|x\rangle = \frac{1}{\sqrt{2^n}} \sum_{y=0}^{2^n-1} \omega^{xy} |y\rangle \qquad \omega = e^{2\pi i/2^n}$$

2^n iterations $\rightarrow$ accuracy $\approx 2^{-n}$. cf time-energy/freq uncertainty

What about input

$$\sum_\alpha a_\alpha |v_\alpha\rangle \quad \text{or} \quad \sum_\alpha |v_\alpha\rangle \otimes |w_\alpha\rangle \,?$$

measuring θ collapses to $\theta_\alpha \approx \theta$

depends on accuracy

Application: Factoring

$$M_a|x\rangle = |ax \bmod N\rangle$$

r = min pos int s.t. $a^r \equiv 1 \bmod N$

$\Rightarrow M_a^r = I$ let $\omega = e^{2\pi i/r}$

$$|v_\alpha\rangle = \frac{1}{\sqrt{r}} \sum_{y=0}^{r-1} \omega^{-\alpha y} |a^y \bmod N\rangle$$

$$= \frac{1}{\sqrt{r}} \left(|1\rangle + \omega^{-\alpha}|a\rangle + \omega^{-2\alpha}|a^2\rangle + \dots \right)$$

$$M_a|v_\alpha\rangle = \omega^\alpha |v_\alpha\rangle$$

$|1\rangle = \frac{1}{\sqrt{r}} \sum_{\alpha=0}^{r-1} |v_\alpha\rangle$ phase est yields $\frac{\alpha}{r}$

specifically $\frac{2^n \alpha}{r}$. classical algos can find r and factors of N

Application: Linear systems (HHL)

Given Hermitian A and $|b\rangle$ find $|x\rangle$

s.t. $A|x\rangle = |b\rangle$

$|x\rangle = \sum_\lambda x_\lambda |a_\lambda\rangle$ ← eigenstates of A

Suppose $\frac{1}{K} \leq \lambda \leq 1$

↓

$$\sum_\lambda x_\lambda |a_\lambda\rangle |\tilde{\lambda}\rangle \rightarrow \sum_\lambda x_\lambda |a_\lambda\rangle |\tilde{\lambda}\rangle \otimes \left(\sqrt{1-\tilde{\lambda}^2}\,|0\rangle + \tilde{\lambda}|1\rangle\right)$$

$$\sum_\lambda x_\lambda |a_\lambda\rangle \otimes \left(\sqrt{1-\left(\frac{K}{\lambda}\right)^2}\,|0\rangle + \frac{K}{\lambda}|1\rangle\right) \rightarrow \approx \sum_\lambda \frac{x_\lambda}{\lambda} |a_\lambda\rangle$$ w/ success prob $\approx \frac{1}{K^2}$

can be improved to $O(K s n \log(1/\epsilon))$

H simulation algorithms

Trotter
$$e^{iAt} e^{iBt} = \left(I + iAt - \frac{A^2 t^2}{2} + \dots\right)\left(I + iBt - \frac{B^2 t^2}{2} + \dots\right)$$

$$= I + i(A+B)t - \frac{t^2}{2}(A^2 + B^2 + 2AB) + O(t^3)$$

c.f. $$e^{i(A+B)t} = I + i(A+B)t - \frac{t^2}{2}(A^2 + B^2 + AB + BA) + O(t^3)$$

$$e^{i(A+B)t} - e^{iAt} e^{iBt} = \frac{t^2}{2}[A,B] + O(t^3)$$

$$\left(e^{\frac{iAt}{r}} e^{\frac{iBt}{r}}\right)^r - e^{i(A+B)t} = O\left(\frac{t^2}{r}\right)$$

$r = \frac{t^2}{\epsilon}$ suffices

<u>operator norm</u> is how we measure error

$$\|A\| = \max \{ \| A|\psi\rangle \| : \| |\psi\rangle \| = 1 \}$$
$$= \max \{ |\lambda| : \lambda \text{ an eigenvalue of } A \}$$ when A is diagonalizable

<u>unitary invariance</u>
$$\|UA\| = \|AU\| = \|A\|$$

implies "hybrid inequality"
$$\|U_1 \cdots U_T - V_1 \cdots V_T\| \le \|U_1 - V_1\| + \|U_2 - V_2\| + \dots + \|U_T - V_T\|$$

Higher-order Trotter-Suzuki formulas

e.g.
$$e^{\frac{iAt}{2}} e^{iBt} e^{\frac{iAt}{2}} = I + i(A+B)t - t^2\left(\frac{A^2}{2\cdot 4} + \frac{B^2}{2} + \frac{A^2}{2\cdot 4} + \frac{AB}{2} + \frac{BA}{2} + \frac{A^2}{4}\right) + O(t^3)$$
$$= I + i(A+B)t - \frac{t^2}{2}(A+B)^2 + O(t^3)$$

Divide into r intervals $\Rightarrow$ error $O(t^3/r^2)$ $\Rightarrow$ $r \sim \frac{t^{3/2}}{\epsilon^{1/2}}$

$O(t^p)$ error scaling is possible with a sequence of length $O(5^p)$ $\Rightarrow r \sim \frac{5^p t^{1+1/p}}{\epsilon^{1/p}}$

Almost matches the $O(t)$ time used by the system itself.

Applications of Hamiltonian Simulation

Ground-state energy estimation

$H|\psi_E\rangle = E|\psi_E\rangle$ are energy eigenstates

The eigenvalues $\{E\}$ are called the spectrum of H
the lowest eigenvalue E_0 is the ground-state energy
$H|\psi_0\rangle = E_0|\psi_0\rangle$ $\quad|\psi_0\rangle$ is the ground state.

Suppose we have a way to prepare a state $|\psi\rangle$
with $|\langle\psi_0|\psi\rangle|^2 \geq \gamma$
Apply phase estimation to this with $U = \exp\left(\frac{-iHT}{2^m}\right)$
and m bits of precision.
This requires performing e^{-iHt} for $0 \leq t \leq T$
and measures energy up to resolution $\sim 1/T$
This can resolve $|\psi_0\rangle$ if the gap $g = E_1 - E_0$ is $\gg 1/T$
Prob[obtain E_0] $= \gamma$ so we need $1/\gamma$ repetitions
or $O(1/\sqrt{\gamma})$ with Grover, a.k.a. amplitude amplification

Preparing ground states?

Even the classical version is hard. (NP-complete)
e.g. max-3-SAT $\cong$ finding ground state energy of a Hamiltonian of the form $|011\rangle\langle011|_{356} + |001\rangle\langle001|_{247} + \ldots$
$(x_3 \vee \bar{x}_5 \vee \bar{x}_6) \wedge (x_2 \vee x_4 \vee \bar{x}_7)$

For general spin Hamiltonians it is QMA-Complete
to estimate the ground-state energy to precision $\sim 1/\mathrm{poly}(n)$
we won't define QMA, but it's like a quantum version of NP.

This means we need heuristic algorithms.

One classical heuristic for minimizing a black-box function $f: \{0,1\}^n \to \mathbb{R}$ is simulated annealing
There is a temperature T that starts high and slowly decreases.
The goal is to maintain samples from p_T with $p_T(x) = \frac{e^{-f(x)/T}}{\sum_y e^{-f(y)/T}}$

Algorithm

Given x_t, choose a nearby trial state y, say with a random bit flip
set $x_{t+1} = \begin{cases} y & \text{with prob } \min\left(1, \frac{p_T(y)}{p_T(x_t)}\right) \\ x_t & \text{with prob } 1 - \min(1, p_T(y)/p_T(x_t)) \end{cases}$

Quantum adiabatic algorithm

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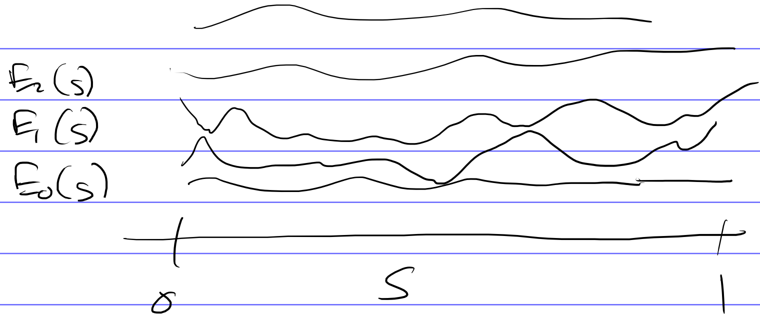
$$H(s) = (1-s)H_0 + s H_1$$



Start in ground state of H_0 $|\psi_0(0)\rangle$

Evolve according to S.E. $\frac{d}{dt}|\psi\rangle = -iH(\frac{t}{T})|\psi\rangle$ with $s = t/T$

Adiabatic thm

If $T \gg \frac{1}{g^2}$ $(g = \min_s E_1(s) - E_0(s))$ then we end in state $|\psi_0(1)\rangle$

Examples

$H_0 = -\sum_{i=1}^n X_i$ $H_1 = \text{diag}(f) = \sum_{z\in\{0,1\}^n} f(z)\, |z\rangle\langle z|$

ground state $|+\rangle^{\otimes n}$ $|z\rangle$ with $z = \arg\min f(z)$

$\langle\psi| H_0 |\psi\rangle = -n + \sum_{i=1}^n \langle\psi| I - X_i |\psi\rangle$ $I - X_i = |-\rangle\langle -|_i$

$= -n + \sum_{x\in\{0,1\}^n} |\psi_x - \psi_{x+e_i}|^2$ $\psi_x = \langle x|\psi\rangle$

minimizing this tries to make ψ_x close at nearby sites, i.e. amplitude is spread out

Could also apply to searching for the quantum ground state with $H_0 = T$ and $H_1 = T+V$

Decoupling

Trotterization is also helpful for building QCs

say $H = \underbrace{\sum_{i=1}^{n-1} Z_i Z_{i+1}}_{\text{always-on interactions}} + \underbrace{\sum_{i=1}^n f_i(t) X_i}_{\text{controls}}$

let $f_i(t) = \frac{\pi}{2}\delta_i(t)$. yields instantaneous X gate

$X_i (Z_i Z_{i+1}) X_i = -Z_i Z_{i+1}$ Lets us selectively turn off interactions

$X_i e^{-iZ_iZ_{i+1}t} X_i = e^{iZ_iZ_{i+1}t}$

Quantum Signal Processing — a unifying paradigm

$$R_j(\theta) = \exp\left(-i\frac{\theta}{2}\sigma_j\right)$$

$$U_{\vec\phi}(\theta) = R_z(\phi_d) R_x(\theta) R_z(\phi_{d-1}) \cdots R_x(\theta) R_z(\phi_0)$$

$$R_x(\theta) = \begin{pmatrix} \cos\theta/2 & -i\sin\theta/2 \\ -i\sin\theta/2 & \cos\theta/2 \end{pmatrix} = \begin{pmatrix} a & i\sqrt{1-a^2} \\ i\sqrt{1-a^2} & a \end{pmatrix}$$

if $\theta = -2\cos^{-1}(a)$

$$U_{\vec\phi}(\theta) = \begin{pmatrix} P(a) & iQ(a)\sqrt{1-a^2} \\ iQ^*(a)\sqrt{1-a^2} & P^*(a) \end{pmatrix}$$

$\deg P \le d \qquad \deg Q \le d-1$

parity of $P = d \bmod 2$ $\quad$ of $Q = d-1 \bmod 2$

$|P|^2 + (1-a^2)|Q|^2 = 1$

Application

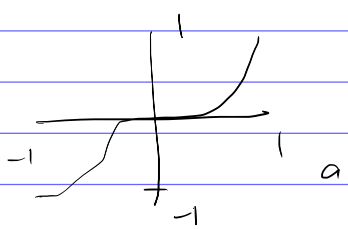
robust rotation, aka composite pulses

want $R_x(\pi)$ $\quad$ get $M_\phi(\pi+\delta) = e^{i\phi Z} e^{i(\pi+\delta)X} e^{-i\phi Z}$

ϕ under our control, δ is not, although it is a systematic error

$a = \cos\left(\frac{\pi+\delta}{2}\right) = \sin(\delta/2)$ $\quad$ want $a=0$

find $P(a)$ like this $\quad$ $P(\delta) \sim \delta^d$



Appl Grover

no marked elt	:	$\theta = 0$	$a = 1$
≥ 1 marked elt	:	$\theta \gtrsim 1/\sqrt{N}$	$a = 1 - 1/N$

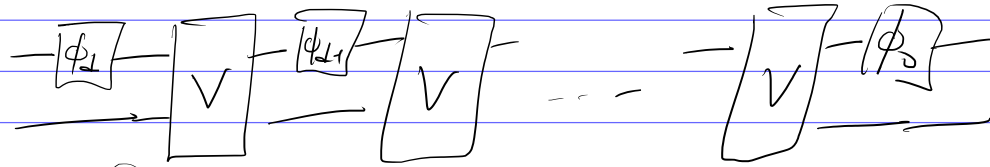
$P(1) = 1 \gg P(1 - 1/N)$?

$P(a) = a^N$ works — no better than classical

$P(a) = \text{Chebyshev}_{\sqrt{N}}(a) = \cos(\sqrt{N}\arccos(a))$

eigenvalue transform

$$V = \begin{pmatrix} H & \sqrt{I-H^2} \\ \sqrt{I-H^2} & -H \end{pmatrix} \qquad H = \sum_\lambda \lambda |u_\lambda\rangle\langle u_\lambda|$$

$U_{\vec{\phi}} =$ 

$$V|0\rangle|u_\lambda\rangle = \left(\lambda|0\rangle + \sqrt{1-\lambda^2}\,|1\rangle\right)|u_\lambda\rangle$$

$V\,|1\rangle\,|u_\lambda\rangle \ldots$

$$\Rightarrow \quad V = \sum_\lambda \begin{pmatrix} \lambda & \sqrt{1-\lambda^2} \\ \sqrt{1-\lambda^2} & -\lambda \end{pmatrix} \otimes |u_\lambda\rangle\langle u_\lambda|$$

$$U_{\vec{\phi}} = \sum_\lambda \begin{pmatrix} P(\lambda) & \cdot \\ \cdot & \cdot \end{pmatrix} \otimes |u_\lambda\rangle\langle u_\lambda|$$

$$= \begin{pmatrix} P(H) & \cdot \\ \cdot & \cdot \end{pmatrix}$$

$P(H) \approx e^{-iHt} \Rightarrow O(t\,\|H\| \log \frac{1}{\epsilon})$ iterations

$P(H) \approx H^{-1} \qquad O(\kappa \log 1/\epsilon)$

$\vdots$

QSVT q. singular value transform