Examples

Trivial code $S = \langle Z_{k+1}, Z_{k+2}, \dots, Z_n \rangle$ k qubits

$N(S) = \langle X_1, Z_1, \dots, X_k, Z_k, S \rangle$ $C = \{ |\psi\rangle \otimes |0\rangle^{\otimes n-k} \}$

Repetition code $S = \langle Z_1 Z_2, Z_2 Z_3 \rangle$

$N(S) = \langle \underbrace{X_1 X_2 X_3}_{\bar{X}}, \underbrace{Z_1}_{\bar{Z}}, S \rangle$

Shor code $S = \langle Z_1Z_2, Z_2Z_3, Z_4Z_5, Z_5Z_6, Z_7Z_8, Z_8Z_9, X_1X_2X_3X_4X_5X_6, X_4X_5X_6X_7X_8X_9 \rangle$

$N(S) = \langle \bar{X} = X_1X_2X_3, \bar{Z} = Z_1Z_4Z_7, S \rangle$

<u>When do Paulis commute?</u> $\Rightarrow$ symplectic $\mathbb{F}_2$ inner product

$p \in P_n$ $p = (-1)^c X^a Z^b$ $q = (-1)^{c'} X^{a'} Z^{b'}$

$pq = (-1)^{c+c'} X^a \underbrace{Z^b X^{a'}}_{} Z^{b'}$

$= Z^{b_1} X^{a_1'} \otimes \dots \otimes Z^{b_n} X^{a_n'}$

$= (-1)^{a_1' b_1} X^{a_1'} Z^{b_1} \otimes \dots$

$= (-1)^{a' \cdot b} X^{a'} Z^b$

$= (-1)^{c+c'+a'\cdot b} X^{a+a'} Z^{b+b'}$

$qp = (-1)^{c+c'+a\cdot b'} X^{a+a'} Z^{b+b'}$

$pq = (-1)^{a'\cdot b + b'\cdot a} qp$

$a'\cdot b + b'\cdot a = \begin{pmatrix} a \\ b \end{pmatrix}^T \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}^T \Omega \begin{pmatrix} a \\ b \end{pmatrix} = \left\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix} \right\rangle_{\text{symplectic}}$

$S \subset P_n$ $\underbrace{L(S) \subset \mathbb{F}_2^{2n}}_{\text{subspace of dim } n}$ $L(N(S)) = L(S)^\perp$

Clifford gp: $U X^a Z^b U^\dagger = \pm X^{a'} Z^{b'}$ $\begin{pmatrix} a' \\ b' \end{pmatrix} = M \begin{pmatrix} a \\ b \end{pmatrix}$

$M^T \Omega M = \Omega$ "symplectic matrix"

corollary Any stabilizer code is equivalent (via Clifford) to the trivial code

$S = \langle s_1, \dots s_{n-k} \rangle \qquad Z(S) = \langle S, \bar{X}_1, \dots, \bar{X}_k, \bar{Z}_1, \dots, \bar{Z}_k \rangle$

$\exists\ U$ (Clifford) s.t. $U s_i U^\dagger = Z_i$

$U \bar{Z}_i U^\dagger = Z_{n-k+i}$

$U \bar{X}_i U^\dagger = X_{n-k+i}$

CSS codes $S = \{ Z^a X^b : a \in C_1^\perp, b \in C_2 \}$

$C(S) = \{ |x + C_2\rangle : x \in C_1 \} \qquad C_2 \subset C_1$

logical operations $X^a : a \in C_1 / C_2$

$Z^b : b \in C_2^\perp / C_1^\perp$

example $[[7,1,3]]$ Steane code

$S = \langle X1X1X1X, \ XX11XX1, \ XXXX111,$
$Z1Z1Z1Z, \ ZZ11ZZ1, \ ZZZZ111 \rangle$

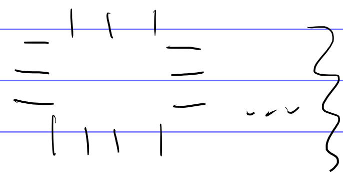
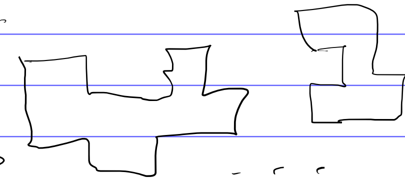
$C(S) = \langle S, \ \bar{X} = X^{\otimes 7}, \ \bar{Z} = Z^{\otimes 7} \rangle$

$N(S)$ includes $H^{\otimes 7} = \bar{H}$

For 2 copies $\overline{Cnot} = Cnot^{\otimes 7}$

TORIC CODE / homology picture of CSS codes

$$A_v = \prod_{e\sim v} X_e \qquad\qquad B_f = \prod_{e\sim f} Z_e$$

$\langle \{A_v\} \rangle = \{$  $\}$ $\qquad \langle \{B_f\} \rangle = \{$  $\}$

$N(S)/S \cong \langle \underset{X}{|||||||}, \underset{X}{\equiv}, \underset{Z}{-----}, \underset{Z}{\vdots} \rangle$

$$\mathbb{F}_2^S = \{\varphi : S \to \mathbb{F}_2\} \cong \mathbb{F}_2^{|S|}$$

$$\mathbb{F}_2^E \cong \mathbb{F}_2^n \qquad n = \#\text{qubits} = 2L^2$$

$$\mathbb{F}_2^F \xrightarrow{\partial_2} \mathbb{F}_2^E \xrightarrow{\partial_1} \mathbb{F}_2^V$$

$\partial_2 \square = \ldots + \ldots + \ldots + \ldots \qquad \partial_1 - = \bullet \cdots + \cdots \bullet$

(over $\mathbb{R}$ these are signed)

$$B_f = Z^{\partial_2 f} \qquad A_v = X^{\partial_1^T v}$$

$$[B_f, A_v] = 0 \iff 0 = \partial_1^T v \cdot \partial_2 f = v^T \partial_1 \partial_2 f$$

$$\boxed{\partial_1 \partial_2 = 0}$$

logical operations

$$Z^b \in N(S) \iff \forall v \sum_{e\sim v} b_e = 0$$

$$\iff \partial_1^T b = 0$$

$$\iff b \in \ker \partial_1^T \iff b = \text{closed loop(s)}$$

$$Z^b \in S \iff Z^b = \prod_{f\in F} B_f^{\beta_f} \iff b = \partial_2 \beta$$

$$\iff b \in \operatorname{Im} \partial_2$$

$$N(S)/S = \ker \partial_1^T / \operatorname{Im} \partial_2 = \{ b + \operatorname{Im} \partial_2 : b \in \ker \partial_1^T \}$$

Knill-Laflamme conditions

given a QECC $C = \mathrm{Im}\, E \subseteq \mathbb{C}^{2^n}$

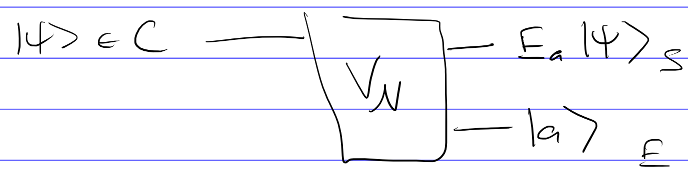
kraus ops $E_1, \dots, E_m$ can be corrected iff

$$P E_a^{+} E_b P = \alpha_{ab} P \qquad \text{where } P = \text{proj } C$$

For simplicity, assume $\alpha_{ab} = \alpha_a \delta_{ab}$

so $E_i C \perp E_j C$ for $i \neq j$

correctable means $\exists$ channel R s.t. $R \circ N \circ E = \mathrm{id}$



$$\mathrm{tr}_S \rightarrow \rho_E = \sum_{a,b} \langle\psi|E_a^{+}E_b|\psi\rangle \, |a\rangle\langle b| = \sum_{ab} c_{ab}|a\rangle\langle b|$$

$$= \langle\psi|P E_a^{+} E_b P|\psi\rangle$$

$$= c_{ab} \text{ indep of } |\psi\rangle$$

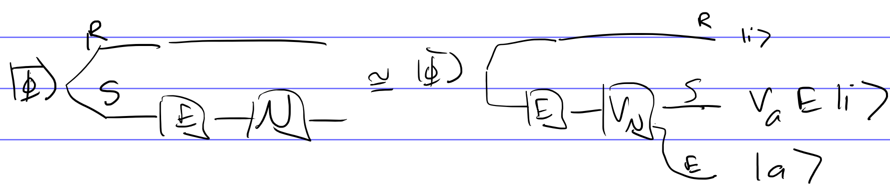
to prove $\exists R$ use two facts

① channel action on $\mathbb{C}^d$ determined by action on half of $|\Phi\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^{d} |i\rangle\otimes|i\rangle$ $\qquad (\mathrm{id}\otimes N)(\Phi) = \frac{1}{d}\sum_{ij} |i\rangle\langle j| \otimes N(|i\rangle\langle j|)$

② $|\psi\rangle_{AB}$ $\quad |\phi\rangle_{AB}$ $\qquad \psi_A = \phi_A \Leftrightarrow \exists\, U_B$ s.t. $|\psi\rangle = (I\otimes U)|\phi\rangle$

① $\Rightarrow$ suffices to consider action on Φ

② $\Rightarrow$ suffices to decouple



$$|\psi\rangle_{RSE} = \sum_{i,a} |i\rangle_R V_a E|i\rangle_S |a\rangle_E$$

$$\psi_{RE} = \psi_R \otimes \psi_E$$

trace out S

$$\frac{1}{d}\sum_{i,j} |i\rangle\langle j| \otimes \sum_{a,b} |a\rangle\langle b| \; \langle j|\underbrace{E^{+}V_b^{+}V_a E}_{P\,\alpha_{ba}}|i\rangle = \frac{I}{d}\otimes\rho_E$$

$$\alpha_{ba}\,\delta_{ij}$$

$\psi_R \otimes \psi_E$ purified by $|\psi\rangle_{RSE}$

and by $|\Phi\rangle_{RS_1} \otimes |\gamma\rangle_{S_2E}$

$\exists\, U_{S\to S S_2}$ s.t.

$$(I_{RE}\otimes U)|\psi\rangle = |\Phi\rangle|\gamma\rangle$$

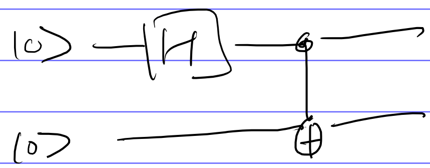
FTQC

1 discrete gate set : Clifford + T

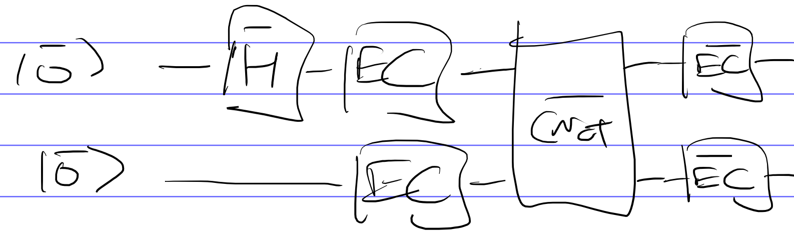
2 never decode

3 don't propagate errors within code blocks

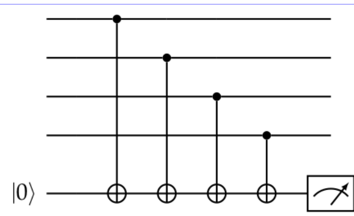
4 FT error correction



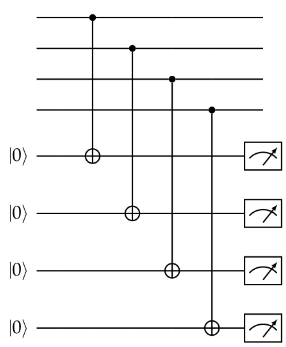
$\Rightarrow$



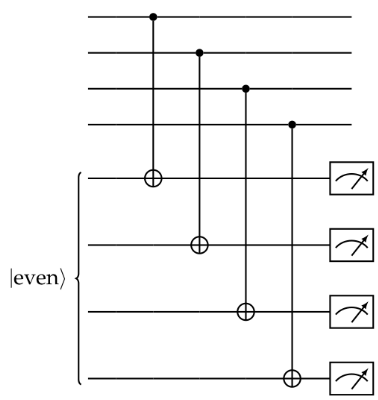
FT measurement :
$Z_1 Z_2 Z_3 Z_4$



not FT



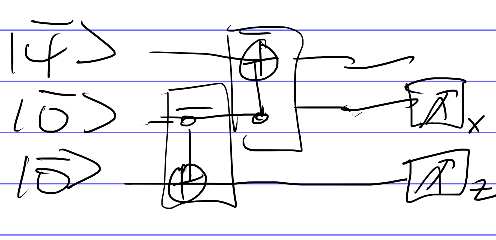
Also no good



Shor

$|\text{even}\rangle = \frac{1}{\sqrt{8}} \sum_{|x| \text{ even}} |x\rangle$

Another choice is Steane



Transversal gates

Steane $\bar{X} = X^{\otimes 7}$ $\quad \bar{Z} = Z^{\otimes 7}$ $\quad \bar{H} = H^{\otimes 7}$

$\overline{\text{Cnot}} = \text{Cnot}^{\otimes 7}$

These generate Clifford group. not universal

need a non-Clifford gate, like $T = \exp(i\frac{\pi}{8}Z)$

given $\bar{T}|\bar{+}\rangle$ can perform $\bar{T}$ by teleportation