Exercise sheet 6

1. **An area law for the mutual information**

   Consider a local Hamiltonian $H = \sum_{(i,j)\in E} h_{i,j}$ where E is a collection of edges defining a graph over a vertex set V and $h_{i,j}$ is a term acting on the qudits at sites i and j. Partition V into $A, \bar{A}$ and write H as

   $$H = H_A + H_{\bar{A}} + H_\partial, \tag{1}$$

   where H_A (resp. $H_{\bar{A}}$) are the terms acting entirely within $A, \bar{A}$ and H_∂ is the sum of all the interactions between A and $\bar{A}$, i.e. either $i \in A, j \in \bar{A}$ or vice versa.

   Let $\sigma := \frac{e^{-H/T}}{\mathrm{tr}\left[e^{-H/T}\right]}$ be the Gibbs state.

   (a) Prove that

   $$I(A;\bar{A})_\sigma \leq 2\frac{\|H_\partial\|}{T} \tag{2}$$

   Note: This factor of 2 was not in the first version of the pset.

   (b) What does this tell us about the amount of entanglement between A and $\bar{A}$ in the ground state? If $T = 0$ then σ is the ground state (or mixture over all ground states) but then eq. (2) is vacuous. However, suppose we further assume a bound on the density of states. Specifically, that the ground state energy is E_0 and that the number of states of energy $\leq E_0 + k$ is $\leq n^k$. Show how this yields a nontrivial bound on the ground-state entanglement entropy.

2. **Area law correction in the surface code**

   This problem is optional. But I hope you find it tempting!

   The surface (or toric) code is defined on a lattice with qubits on each edge, and stabilizer generators

   $$A_s = \prod_{e\sim s} X_e \qquad\qquad B_p = \prod_{e\sim p} Z_e \tag{3}$$

   Here s refers to a "site" (or vertex) and $e \sim s$ means that e is one of the four edges touching this site. These four edges make a star. Next p refers to a "plaquette" (or square) and $e \sim p$ refers to the four edge bordering this plaquette. This is illustrated in fig. 1(a).

   For simplicity, we will consider the surface code with smooth boundaries, or else on the sphere, so that there are no logical qubits. This means that there is a unique state $|\psi\rangle$ stabilized by all the $\{A_s\}$ and $\{B_p\}$. (The story is similar for the case of the torus or rough boundaries when there are some logical qubits but we wish to avoid those complications for now.)

   Let A be an subregion of size $k \times k$, illustrated in fig. 1(b). Compute $S(\psi_A)$ as a function of k. Your answer should be of the form $\alpha k - \gamma$. The term γ is known as

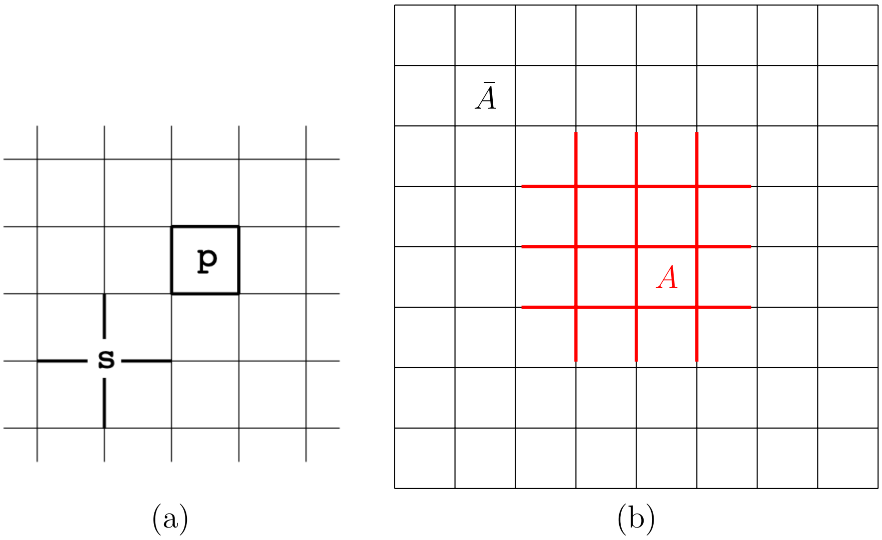

Figure 1: (a) Star A_s and plaquette B_p operators. (b) A is a subregion of size $k \times k$. Here $k = 4$.

the topological entanglement entropy and was introduced in hep-th/0510092. (You do not need anything from that paper to solve this problem.)

As a hint, a stabilizer state $|S\rangle$ is defined in terms of a maximal stabilizer subgroup S of the Pauli group $P_n := \pm\{I, X, iY, Z\}^{\otimes n}$. We require that $-1 \in S$, S is abelian and $|S| = 2^n$, and then have

$$|S\rangle\langle S| = |S|^{-1} \sum_{s \in S} s = \prod_{i=1}^{n} \frac{I + g_i}{2}, \tag{4}$$

where $g_1, \ldots, g_n$ generates S. If instead $|S| = 2^m$ for $m \leq n$ then in general we obtain the mixed state

$$\rho_S := |S|^{-1} \sum_{s \in S} s = \prod_{i=1}^{m} \frac{I + g_i}{2}. \tag{5}$$

This has 2^{n-m} eigenvalues, each equal to 2^{m-n}.