Exercise sheet 3

1. **Error correction with the quantum Hamming code**

   Suppose I encode a qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ with the 7-qubit Hamming code, which is CSS($C_1 : C_2$) with codes C_1, C_2 having generator matrices G, H respectively:

$$G = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \qquad H = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

   The encoded state is stored in a noisy memory, and two errors occur (this is one more error than the code is designed to correct). When you decode the state with errors, what one-qubit state will you obtain if:

   (a) There is an $X_3 Z_6$ error, i.e. a σ_x error on qubit 3 and a σ_z error on qubit 6,

   (b) There is an $X_3 Y_6$ error.

2. **A [4,2,2] quantum code** There is a CSS code with distance 2 (which thus can detect one error, but not correct any) with the following codes C_1 and C_2:

$$C_1 = \{0000, 0011, 0101, 1001, 0110, 1010, 1100, 1111\}, \qquad C_2 = \{0000, 1111\}.$$

   (a) Write down the four codewords of this code.

   For the next three questions, you don't have to list what happens to all four codewords if you can give a systematic description of the effects of the operations. In particular, if they correspond to a logical operation (i.e. a unitary transformation within the codespace), then you can just identify the logical operation.

   (b) Suppose you apply X to qubits 1 and 2 of this code (assume the qubits are labeled 1, 2, 3, 4). Does this operation take codewords to other states in the code? What states does it take the four codewords to?

   (c) Suppose you apply a Z to qubits 1 and 3 of this code. Does this operation take codewords to codewords? What states does it take the four codewords to?

   (d) Suppose you apply H to all four qubits. Does this operation take the four codewords to states in the code? What states does it take the four codewords to?