Exercise sheet 2

1. **Generalized Shor code**

   Suppose k and ℓ are two odd numbers. There is a generalization of the nine-qubit code to a $k\ell$-qubit code that has the following codewords:

$$|0\rangle_L = \frac{1}{2^{\ell/2}}\Big(|\underbrace{000\ldots0}_{k}\rangle + |\underbrace{111\ldots1}_{k}\rangle\Big)^{\otimes\ell},$$
$$|1\rangle_L = \frac{1}{2^{\ell/2}}\Big(|\underbrace{000\ldots0}_{k}\rangle - |\underbrace{111\ldots1}_{k}\rangle\Big)^{\otimes\ell}.$$

   How many bit errors can this code correct? How many phase errors can this code correct? How many syndrome bits do you need to measure to correct the bit errors? How many syndrome bits do you need to measure to correct the phase errors? (For the last two questions, I am asking how many bits are in the syndrome you compute, not how many bits you need to XOR to find each bit of the syndrome.)

2. **Hamming balls and random linear codes** The Hamming weight of $x \in \{0,1\}^n$ is denoted $|x|$ and equals the number of positions in x that are equal to 1. Use Stirling's forrmula $\log(N!) \approx N\log(N/e)$ to show that the number of x with Hamming weight $\leq pn$ is $\approx \exp(nH_2(p))$ where $H_2(p) = -p\log(p) - (1-p)\log(1-p)$. Here we assume $0 \leq p \leq 1/2$, we consider log and exp to be base-2 and we neglect poly(n) factors in our approximation.