Exercise sheet 1

1. **Pure states** Let ρ be a finite-dimensional density matrix. We say that ρ is a pure state if $\rho = |\psi\rangle\langle\psi|$ for some $|\psi\rangle$. Prove that $\operatorname{tr}(\rho^2) = 1$ if and only if ρ is pure.

2. **Complementary channel** In class, we wrote down an isometry $V : A \mapsto B \otimes E$ such that $\operatorname{tr}_E[V\rho V^\dagger] = \sum_i \operatorname{tr}[M_i\rho]\, |i\rangle\langle i|$. Suppose that we trace out system B instead of system E, i.e. we consider $\operatorname{tr}_B[V\rho V^\dagger]$. Write down the resulting channel as a function of ρ and the $\{M_i\}$. What is an operational interpretation of this channel? What happens if $M_i = |i\rangle\langle i|$?

3. **Positive but not completely positive** Let's check why quantum operations need to be completely positive and not only positive. Let T denote the transpose operation, i.e. $T(|i\rangle\langle j|) = |j\rangle\langle i|$ in the standard basis. Show that T is a positive but not completely positive operation. In other words,show that if $X \succeq 0$ then $T(X) \succeq 0$ and find some quantum state ρ such that $(\mathrm{id} \otimes T)(\rho) \not\succeq 0$.

4. **Trace distance** Suppose that you are given one of two possible d-dimensional states σ_1 or σ_2, with probabilities p_1 and $p_2 = 1 - p_1$ respectively. Your task is to perform a two-outcome measurement and then try to guess which state you had been given, minimizing the probability of error.

   If the measurement elements are nonnegative Hermitian matrices $M_1 = M$ and $M_2 = I - M$ then the probability of guessing wrong is

   $$P_{err} = p_1 \operatorname{tr}(\sigma_1 M_2) + p_2 \operatorname{tr}(\sigma_2 M_1). \tag{1}$$

   (a) Show that

   $$P_{err} = p_1 + \operatorname{tr}[M\Delta], \tag{2}$$

   where $\Delta = p_2\sigma_2 - p_1\sigma_1$.

   (b) *Math interlude: duality of S_1 and S_∞ norms.* Consider Hermitian matrices X, Λ. Let $\|X\|_1$ denote the Schatten 1-norm, i.e. the sum of the absolute values of the eigenvalues of X, and let $\|\Lambda\|_\infty$ denote the Schatten ∞-norm, i.e. the maximum absolute value of the eigenvalues of Λ. Prove that

   $$\|X\|_1 = \max_{\|\Lambda\|_\infty \le 1} \operatorname{tr}[X\Lambda] \tag{3}$$

   $$\|\Lambda\|_\infty = \max_{\|X\|_1 \le 1} \operatorname{tr}[X\Lambda] \tag{4}$$

   (These relations say that the Schatten 1-norm and the Schatten ∞-norm are *dual* to each other.)

   (c) Find the PSD operator M that minimizes P_{err}. Show that the resulting error probability is $P_{err,opt} = p_1 - \sum_{i:\lambda_i<0} |\lambda_i|$, where $\{\lambda_i\}$ are the eigenvalues of Δ.

   (d) Evaluate $P_{err,opt}$ in the following cases:

      i. $p_1 = 1$, $p_2 = 0$ and σ_1, σ_2 are arbitrary.

ii. $p_1 \geq p_2 \geq 0$ are arbitrary (subject to $p_1 + p_2 = 1$) and $\sigma_1 = \sigma_2$.

iii. $p_1 = p_2 = 1/2$ and σ_1, σ_2 are arbitrary. Express your answer in terms of $\|\sigma_1 - \sigma_2\|_1$.

iv. $p_1 = p_2 = 1/2$, $\sigma_1 = |\psi_1\rangle\langle\psi_1|$, $\sigma_2 = |\psi_2\rangle\langle\psi_2|$. Express your answer in terms of $|\langle\psi_1|\psi_2\rangle|^2$.